The pricing mechanism to the buyer with a budget constraint and an indirect mechanism

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Abstract

The present article considers the situation in which the buyer's taste and budget are his private information. In this multi-dimensional setting, we study the optimal mechanism through a canonical mechanism in the traditional one-dimensional context: a function of one variable, the buyer's taste. In our multi-dimensional context, however, this is an indirect mechanism. We investigate the effectiveness and limit of this indirect mechanism in the framework of the revelation principle.

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1 Introduction

We consider the optimal selling mechanism with the budget-constrained buyer who has his taste and budget as private information. This is an adverse selection problem in two dimensions. It is widely known that the multi-dimensional advese selection problem involves quite a few technical complications(see Armstrong (1996), Rochet and Chone (1998)). One method for circumventing the difficulties is to reduce the dimension of private information(Rochet and Stole (2003)).

To deal with our problem, Che and Gale (2000) resorted to the reduction of dimensions by means of the nonlinear pricing, an indirect mechanism. The present article investigates the effectiveness of another indirect mechanism, i.e. the canonical one dimensional mechanism—a map from the taste space to the quality and price space—as an optimal multi-dimensional mechanism. Since our pricing mechanism does not include the buyer's budget, it may well happen that the buyer does not have a sufficient budget to buy a commodity destined for his taste. We examine how this mechanism functions.

We may see some similarity between the approach of this paper and the one of Rochet and Stole (1997) and Rochet and Stole (2002). They studied the optimal selling mechanism of duopolistic sellers in Hotelling's environment where buyers have private information of their taste and distance from the sellers. Since the private information is two-dimensional, as a direct mechanism approach it will be natural to consider the scenario that the sellers design an incentive scheme as a map from the taste and distance to the quality-price pair. Instead, Rochet and Stole (1997) took an alternative approach, in which the sellers make a price scheme which associates only the buyers' taste with a quality-price pair while regarding the distance as a random variable. The weak mechanism of the present paper, in the similar manner, takes in only the buyer's taste.

The revelation principle asserts that any equilibrium allocation realised by a mechanism can be effecuated by a direct incentive compatible mechanism. Indirect as it is, our mechanism might risk some generality. We examine when equivalence is guaranteed between our mechanism and a direct mechanism.

In the next section, the model is presented. Section 3 formally describe the seller's mechanism design. Section 4 treats, as a reference case, the selling mechanism with no budget constraint. The case of the budget-constrained buyer is dealt with in Section 5. In Section 6, we perform a comparison between the weak mechanism design approach and the strong one.

2 The model

There is a seller and a buyer who are both risk-neutral. The seller has one unit of an indivisible commodity to sell of a quality q such that $q \in Q := [0, 1]$. Alternatively, it can be interpreted so that the seller has one unit of a divisible commodity and q is a quantity. The buyer purchases either one unit of the commodity of quality q or none. The seller values the commodity at zero. The buyer has a *taste* t for the commodity, which takes a value in R_+ where R_+ is the set of non-negative real numbers. Likewise, the buyer has a budget w, which takes a value in R_+. The couple (t, w) is distributed according to the density $g(t, w)$ continuous and positive on its support, which is $T \times W$ where $T := [0, \bar{t}]$ and $W := [0, \overline{w}]$.[1] (t, w) is the buyer's private information. The seller only knows the density $g(t, w)$. We shall call the pair (t, w) the buyer's *type* from now onwards.

Let us denote the function derived from $g(t, w)$ by $G(t, w)$:

$$G(t, w) := \int_0^w g(t, x)dx.$$

$G(t, w)$ is the probability that the buyer of taste t has a budget smaller than w.

The buyer's utility function is assumed to be of quasi-linear form. Thus, buying

[1]Naturally, these are assumed non-empty.

quality q and paying price p, taste t buyer obtains utility, $tq - p$.

We assume

$$\bar{t} \leq \overline{w}.$$

The buyer of the highest taste $\bar{t}$ obtains the utility $\bar{t} - p$ for the highest quality $q = 1$ and the price p. The assumption indicates that with the highest budget, he can pay the highest price $\bar{t}$, for which he obtains zero utility(for the price higher than that, buyer $\bar{t}$ chooses not to purchase).

3 The mechanism

In adverse selection literature, there are two approaches, one of which is a direct mechanism approach and the other a non-linear pricing(indirect mechanism)approach. By extending Wilson (1993), let us sketch the latter approach first.

The non-linear price scheme $\tau(q)$ is defined as a map of the space Q to the price space R. It is assumed that $\tau(q)$ is lower semi-continuous and almost everywhere differentiable. The buyer of type (t, w) purchases a quality which satisfies

$$\max_{q \in \{x | \tau(x) \leq w\}} tq - \tau(q). \tag{1}$$

We can write the measure of the buyers of tastes who purchase a quality higher than q,

$$\mathcal{M}(\tau(q), q) := \text{Prob}\{(t, w) | (\exists x \geq q)(\forall y < q \text{ s.t. } \tau(y) \leq w)\, ty - \tau(y) \leq tx - \tau(x),\ \tau(x) \leq w\}.$$

If the programme (1) is quasi-concave, we obtain

$$\mathcal{M}(\tau(q), q) = \text{Prob}\{(t, w) | t \geq \tau'(q),\ \tau(q) \leq w\} =: M(\tau(q), \tau'(q), q).$$

It follows immediately that

$$M(\tau(q), \tau'(q), q) = \int_{\tau(q)}^{\overline{w}} \int_{\tau'(q)}^{\bar{t}} g(t, w) dw dt.$$

The seller maximises the following expected profits

$$\int_0^1 \tau'(q) M(\tau(q), \tau'(q), q) dq.$$

This depends on τ as well as τ' contrary to the case where there is no budget constraint(see Wilson (1993)) so that we cannot perform pointwise maximization. Che and Gale (2000) resorted to this formalisation to reduce the dimenstion in our two-dimensional problem.

The standard mechanism design approach consists in parameterising the buyer with his type (t, w) and assigns a quality-price pair to each type, what is called a direct mechanism. In this section, we take a weaker approach. The weak mechanism determines the quality and the price only for the buyer's taste. Formally, the *weak mechanism* is defined as a mapping

$$(q, p) : T \to Q \times R.$$

The qualification *weak* is used to avoid any confusion with the other to be introduced in a later section. The weak mechanism is not a direct mechanism; for the domain of the weak mechanism is T while the type space is $T \times W$.

In order to induce the buyer of taste t to choose the pair $(q(t), p(t))$ and divulge his real taste, the mechanism has to satisfy the incentive compatibility constraint

Definition 1. *It is said that a weak mechanism* $(q(t), p(t))$ *satisfies the* weak incentive compatibility constraint*(WIC) if and only if*

$$tq(t) - p(t) \geq tq(\tilde{t}) - p(\tilde{t}) \quad \textit{for any } t, \tilde{t} \in T. \tag{WIC}$$

Moreover, in order to persuade the buyer to participate in the purchase, the seller has to assure him of minimum utility as the participation(or individual rationality) constraint. Normalising the reservation utility to zero, we have the participation constraint[2]:

$$u(t) := tq(t) - p(t) \geq 0 \quad \text{on } T. \tag{WIR}$$

We have defined the WIC mechanism the same way as if there is no budget constraint. Given a WIC mechanism, the buyer of taste t may not be able to choose $(q(t), p(t))$ if he has a limited budget. Section 5 addresses this issue.

As is standard, instead of the quality-price pair $(q(t), p(t))$, we set up our problem with the quality-utility pair $(q(t), u(t))$.

Lemma 1. *If the weak mechanism $(q(t), p(t))$ is WIC, the following conditions are satisfied*[3]*;*

$$u(t) \text{ is absolutely continuous,} \tag{2}$$

$$q(t) \text{ is non-decreasing,} \tag{3}$$

$$q(t) = \dot{u}(t) \qquad a.e. \tag{4}$$

Conversely, given $q(t)$ and $u(t)$ which satisfies (2), (3) *and* (4)*, the WIC mechanism $(q(t), p(t))$ can be constructed, by putting*

$$p(t) = tq(t) - u(t). \tag{5}$$

Proof. See Rochet (1985). □

In the perfect information case where t is observable and there is no budget constraint,

[2] W in WIR stands for weak.
[3] a.e. stands for almost everywhere.

the seller maximises his profit p subject to the participation constraint $tq - p \geq 0$, which leads to the first best efficient allocation, $q = 1$, $p = t$ and $u = 0$.

4 No budget constraint

As a reference case, let us consider the case in which the buyer does not have a budget constraint and thus only the taste is his private information. Then, he can purchase a quality designated for him by a weak mechanism. Since there is no limit in the buyer's budget, we shall leave out the variable w from the density $g(t, w)$ and denote the density of t by $g(t)$ and the distribution function by $G(t)$ in this section. (WIR) can be replaced by $0 \leq u(0)$ in view of (4). The seller, thus, maximises with respect to q and u his expected profit

$$\int_0^{\bar{t}} p(t)g(t)dt = \int_0^{\bar{t}} (tq - u)g(t)dt,$$

subject to $0 \leq q \leq 1$ and the incentive compatibility constraints (2), (3), (4) and the participation constraint $0 \leq u(0)$.

As usual, we ignore the monotonicity of q and make sure that it is indeed satisfied at the end.

Writting the optimal solution (q^*, u^*), we easily obtain that $u^*(0) = 0$ from the transversality condition and also that the Hamiltonian is expressed as

$$H(t, q, u, \lambda) = (tq - u)g + \lambda q = (tq - u)g + (G - 1)q$$

where λ is an absolutely continuous adjoint variable.

Now let us introduce an assumption.

Assumption 1. $g(t)t + G(t) - 1$ *is strictly increasing.*

Proposition 1. *Suppose that Assumption 1 is satisfied. Then, there exists the unique*

$\hat{t} \in (0, \bar{t})$ *such that*

$$g(\hat{t})\hat{t} + G(\hat{t}) - 1 = 0,$$

and the optimal solution and price are as follows,

$$(q^*(t), u^*(t), p^*(t)) = \begin{cases} (0,0,0) & \text{if } t \in [0, \hat{t}], \\ (1, t-\hat{t}, \hat{t}) & \text{if } t \in [\hat{t}, \bar{t}]. \end{cases}$$

Proof. $\frac{\partial H}{\partial q}(0, q, u^*, \lambda) < 0$ and $\frac{\partial H}{\partial q}(\bar{t}, q, u^*, \lambda) > 0$. Therefore, by Assumption 1, there is a unique $\hat{t}$ satisfying $\frac{\partial H}{\partial q}(\hat{t}, q, u^*, \lambda) = 0$. Now the proposition is obvious. □

In comparison with the first best allocation, the usual observation in the asymmetric information problem is noticed: inefficient quality allocation and full rent extraction at the lowest taste, no quality distortion at the highest taste.

The literature on mechanism design(e.g. Maskin and Riley (1984)) asserts that with a slight assumption on the density of the buyer's taste, the seller discriminates a portion of tastes and offers the distinct price-quality option. By contrast, here, all the tastes are offered either zero or the highest quality: hence bunching. In addition, a portion of the lower tastes are excluded from purchase. These features result from the lack of production cost in the present model.

5 Budget constraint

We deal with the optimal weak mechanism with the budget-constrained buyer. We need a few assumptions on the density $g(t, w)$. Except in Section 6, we suppose all the assumptions below to be satisfied.[4]

Assumption 2. *For all* (t, w) *in* $T \times W$, $g(t, w) > 0$.

[4]Assumption 1 is not supposed, however.

Assumption 3. *$g(t, w)$ is continuously differentiable on $T \times W$.*

A main question in this section is what quality the buyer purchases if he cannot afford one designed for his taste on account of his limited budget. Given a WIC mechanism, the buyer of type (t, w) actually purchases the quality-price pair $(q(t), p(t))$ if $p(t) \leq w$. On the contrary, if the price is too high, $p(t) > w$, the buyer is bound to choose another quality within his budget. Type (t, w) buyer's decision upon a purchase results from,

$$\max_{t' \text{ s.t. } p(t') \leq w} tq(t') - p(t').$$

To describe the seller's maximisation program, we have to spell out what quality a buyer purchases when he has a small budget. Let us begin with a straightforward observation.

Lemma 2. *Let a mechanism $(q(t), p(t))$ be WIC. Then $p(t)$ is non-decreasing.*

Proof. Suppose that $(q(t), p(t))$ is WIC. In addition, let us assume, as opposed to the proposition, that $p(t) > p(t')$ for $t \leq t'$. Then we have

$$tq(t) - p(t) \leq tq(t') - p(t) < tq(t') - p(t').$$

The first inequality follows from (3) of Lemma 1. It is obvious that $tq(t) - p(t) < tq(t') - p(t')$ contradicts the WIC condition.

$\square$

Along with (3), the proposition states that a higher quality is coupled with a higher price. Thus, taste t buyer unable to afford the assigned quality is obliged to choose a lower quality for a lower taste.

If the price is continuous, the buyer's quality choice can be expressed in a simple manner as will be seen. The discontinuous price involves complications. Notice, though, that given a WIC mechanism, the price $p(t)$ is almost everywhere differentiable from Lemma 2. With a few conditions on $p(t)$, we obtain the following proposition.

Proposition 2. *Suppose that there is given a WIC mechanism $(q(t), p(t))_{t\in T}$. Suppose that the buyer is of taste-budget (t, w). Further, suppose that p is left-continuous. Finally, suppose that the buyer of that type (t, w) cannot afford the quality intended for him, namely $p(t) > w$.*

If $w < p(0)$, the buyer cannot purchase any quality; that is,

$$w < p(x) \qquad \textit{for all } x \in T.$$

If $p(0) \leq w$, there exists $\mu(w) := \max\{x|p(x) \leq w\}$ and the buyer purchases the quality-price pair $(q(\mu(w)), p(\mu(w)))$; i.e. $\mu(w)$ solves

$$\max_{x \; s.t. \; p(x)\leq w} tq(x) - p(x).$$

Moreover, $\{x|p(x) \leq w\} = [0, \mu(w)]$ and $p(\mu(w)) = \max\{p(x)|p(x) \leq w\}$.

Proof. From Lemma 2, if $w < p(0)$, it follows that $w < p(t)$ for all t. Therefore, the buyer affords no quality.

Now suppose $p(0) \leq w$. Then $\{x|p(x) \leq w\} \neq \emptyset$. Since p is non-decreasing and left continuous, there exists $\mu(w) := \max\{x|p(x) \leq w\}$ and $[0, \mu(w)] = \{x|p(x) \leq w\}$. Again, from the monotonicity of p, it follows that $p(\mu(w)) \geq p(x)$ for $x \in [0, \mu(w)]$.

Let us show that taste t buyer chooses $(q(\mu(w)), p(\mu(w)))$. Suppose $t' < \mu(w)$. Then it suffices to show that $tq(t') - p(t') \leq tq(\mu(w)) - p(\mu(w))$. As is seen above, $t' \in \{t|p(t) \leq w\}$ so that taste t buyer can purchase quality t'.

Now let us suppose $tq(t') - p(t') > tq(\mu(w)) - p(\mu(w))$ and deduce the contradiction. Recall that q is non-decreasing by (3) and that $t' < \mu(w) < t$ since p is non-decreasing.

Then, it follows that

$$0 > t(q(\mu(w)) - q(t')) - (p(\mu(w)) - p(t')) > \mu(w)(q(\mu(w)) - q(t')) - (p(\mu(w)) - p(t')).$$

It is equivalent to

$$\mu(w)q(t') - p(t') > \mu(w)q(\mu(w)) - p(\mu(w)),$$

which is contradictory to the definition of the WIC.

□

The proposition states that if the buyer is not well off enough to buy a quality designed for him, he buys a quality whose price is highest within his budget and the quality is also highest whithin his budget.

Supposing that the price is continuous, a more plain statement can be made. Later on, we shall resort to the following proposition.

Proposition 3. *Let all the assumptions of the previous proposition be satisfied, except that* p *is left-continuous. Instead, we assume that* p *is continuous. If* $p(0) \leq w$, $\mu(w)$ *in Proposition 2 satisfies* $p(\mu(w)) = w$.

Proof. There exists t' such that $p(t') = w$ because p is continuous and $p(0) \leq w < p(t)$. There also exists $m := \max\{t' | p(t') = w\}$ by continuity. Then $\mu(w) \leq m$ by the monotonicity of p and $m \leq \mu(w)$ from the definition of $\mu(w)$.

□

The proposition indicates that when the price is continuous, taste t buyer with an insufficient budget purchases quality $q(\mu(w))$, spending all his budget.

Finally, we turn to the participation constraint. Given a WIC mechanism, taste t buyer wealthy enough purchases $(q(t), p(t))$ and thus it suffices to have $0 \leq u(t)$ as the participation constraint. However, if he is constrained in the budget, he has to choose another quality-price pair. We need therefore the participation constraint which ensures that the buyer participates even if forced to choose another quality-price pair. Since taste t buyer, if not wealthy enough, chooses a pair $(q(t'), p(t'))$ such that $t' < t$, the participation constraint must guarantee that for all t and t' such that $t' \leq t$,

$$0 \leq tq(t') - p(t').$$

It turns out the usual participation constraint $0 \leq u(t)$ assures this.

Lemma 3. *Let the WIC mechanism $(q(t), p(t))$ be given. Then $0 \leq u(t)$ for all t if and only if for all t and t' such that $t' \leq t$,*

$$0 \leq tq(t') - p(t').$$

Proof. Suppose that $0 \leq u(t)$ on T. Then it follows that for t and t' such that $t' \leq t$,

$$0 \leq u(t') = t'q(t') - p(t') \leq tq(t') - p(t').$$

The converse is obvious. □

Let us express the seller's profit by use of Proposition 2. Suppose that there is a WIC mechanism $(q(t), p(t))$ such that $p(t)$ is left-continuous. It follows from the proposition that

$$\operatorname*{arg\,max}_{(q(s),p(s))\,\text{s.t.}\,p(s)\leq w} tq(s) - p(s) = \begin{cases} (0,0) & \text{if } w < p(0), \\ (q(\mu(w)), p(\mu(w))) & \text{if } p(0) \leq w < p(t), \\ (q(t), p(t)) & \text{if } p(t) \leq w, \end{cases} \tag{6}$$

where we have adopted a convention that if $\{p(s) \leq w\}$ is empty, the left side of (6) is equal to $(0,0)$.

The seller obtains the following profit from taste t buyer,

$$\Pi(p(t), w) = \begin{cases} 0 & \text{if } w < p(0), \\ p(\mu(w)) & \text{if } p(0) \leq w \leq p(t), \\ p(t) & \text{if } w \geq p(t). \end{cases} \tag{7}$$

We can now express the optimal mechanism.

Definition 2. *The* optimal *weak mechanism is a weak mechanism which maximises the seller's expected profits and satisfies* (WIC) *and* (WIR)*.*

We suppose that $p(t)$ is continuous from now onward and make use of Proposition 3. By replacing $p(t)$, we write the seller's profit $\int_0^{\bar{t}} \int_0^{\overline{w}} \Pi(p(t), w) g(t, w) dt dw$ as

$$\int_0^{\bar{t}} dt \big(\int_{-u(0)}^{tq(t)-u(t)} g(t, w) w dw + \int_{tq(t)-u(t)}^{\overline{w}} g(t, w)(tq(t) - u(t)) dw \big).$$

By partial integration in the first term, it turns into

$$\begin{aligned} \int_0^{\bar{t}} dt \bigg(G(t, tq(t) - u(t))(tq(t) - u(t)) - \int_0^{tq(t)-u(t)} G(t, w) dw + \\ \big(G(t, \overline{w}) - G(t, tq(t) - u(t)) \big)(tq(t) - u(t)) \bigg) \\ = \int_0^{\bar{t}} dt \big(G(t, \overline{w})(tq - u) - \int_0^{tq(t)-u(t)} G(t, w) dw \big). \end{aligned} \tag{8}$$

The seller maximises this expected profit subject to (2), (3), (4) and $0 \leq u(0)$. The continuity assumption of $p(t)$ can be replaced by that of $q(t)$ owing to (2).

Let us now differentiate the objective function (8) with respect to u. Then we have

$$-G(t, \overline{w}) + G(t, tq - u) \leq 0.$$

The inequality results from $tq - u \leq \bar{t} - u \leq \overline{w}$. It follows that given an admissible pair (q, u), it will increase the value of the objective function to modify u by adding a constant in such a way that $0 \leq u(0)$ binds. Note that doing so does not modify q due to (4). In consequence, we can replace $0 \leq u(0)$ with $u(0) = 0$ as the participation constraint.

Now we state the seller's maximisation program for the optimal weak mechanism;

$$\max_{q,u} \int_0^{\bar{t}} \Big(G(t, \overline{w})(tq - u) - \int_0^{tq-u} G(t, w)dw\Big)dt \tag{9}$$

s. t.

$$0 \leq q \leq 1, \tag{10}$$

$$\dot{u} = q \quad \text{a.e,} \tag{11}$$

$$q \text{ is non-decreasing and continuous,} \tag{12}$$

$$0 = u(0). \tag{13}$$

As is standard, we search for the solution among measurable control variable q's and absolutely continuous state variable u's while first ignoring (12).

However, if we incorporate (12) from the begining, i.e. two conditions $\dot{q} = z$ a.e. and $0 \leq z$ and replace (10) with $0 \leq q(0)$ and $q(\bar{t}) \leq 1$, the problem amounts to that of control variable z and state variables u and q. Since standard optimal control takes a state variable as absolutely continuous, the second condition in (12) is automatically satisfied. And yet this approach does not lead us to obtain an explicit result.

Let us call the above maximisation problem without (12) the *relaxed* problem. Later we will ascertain that the solution of the relaxed problem is indeed that of the original problem. First, the existence of a solution is asserted by an elementary existence theorem.

Proposition 4. *There exists an optimal solution for the relaxed problem.*

Proof. Let us resort to Filippov's existence theorem(see p.314 of Cesari (1983)). From (11) and (13), we know that

$$u(t) = \int_0^t q(s)ds \leq \bar{t}.$$

Since q is non-negative, u is non-decreasing. Thus, $u(t) \in [0, \bar{t}]$, which is compact. Likewise, it is easily verified that all the conditions for the existence theorem are satisfied. □

Let us write the Hamiltonian

$$H(t, q, u, \lambda) = G(t, \overline{w})(tq - u) - \int_0^{tq-u} G(t, w)dw + \lambda q.$$

where λ is an adjoint variable.

If (q^*, u^*) is an optimal pair of the relaxed problem, then there exists an absolutely continuous λ which satisfies the following conditions:

1. For almost every t,

$$H(t, q^*, u^*(t), \lambda(t)) = \max_{q \in [0,1]} H(t, q, u^*(t), \lambda(t)).$$

2. Almost everywhere,

$$\dot{\lambda} = G(t, \overline{w}) - G(t, tq^* - u^*).$$

3. As the transversality conditions,

$$\lambda(\bar{t}) = 0.$$

The Hamiltonian is concave and differentiable in q and we have

$$\frac{\partial H}{\partial q}(t, q, u^*(t), \lambda(t)) = t(G(t, \overline{w}) - G(t, tq - u^*(t))) + \lambda(t).$$

Since $H(t, q, u^*(t), \lambda(t))$ is concave in q, it follows that

$$\frac{\partial H}{\partial q}(t, 0, u^*(t), \lambda(t)) \geq \frac{\partial H}{\partial q}(t, q, u^*(t), \lambda(t)) \geq \frac{\partial H}{\partial q}(t, 1, u^*(t), \lambda(t)).$$

We introduce two crucial assumptions for the rest of the analysis.

Assumption 4. *$tG(t, \overline{w})$ is non-decreasing in t.*

Assumption 5. *For all $(t, w) \in T \times W$,*

$$G(t, \overline{w}) - G(t, w) + t\big(G_1(t, \overline{w}) - G_1(t, w)\big) \geq 0.$$

where $G_1(t, w)$ is the partial differential with respect to the first variable.

These two assumptions are essential to ensure that the ignored condition of the original problem, the monotonicity and continuity of $q(t)$ is satisfied as the solution of the relaxed problem. We assume these throughout the section. The following result is obtained. The proof is relegated to the appendix.

Proposition 5. *The solution of the relaxed problem satisfies* (12) *and thus it is the solution of the original problem.*

The solution is characterised as follows. There exist the unique t' and unique t'' such that $0 < t' < t'' \leq \bar{t}$ and for $t \in (t', t'')$, there is the unique $\hat{q}(t)$ which maximises the Hamiltonian,

$$t(G(t, \overline{w}) - G(t, t\hat{q}(t) - u^*(t))) + \lambda(t) = 0.$$

The solution q^* is uniquely determined such that

$$q^*(t) = \begin{cases} 0 & \textit{if } t \in [0, t'], \\ \hat{q}(t) & \textit{if } t \in (t', t''), \\ 1 & \textit{if } t \in [t'', \bar{t}]. \end{cases}$$

And furthermore, on (t', t'')*,* $q^*(t)$ *is strictly increasing, differentiable and* $0 < q^*(t) < 1$.

The standard characteristics of the asymmetric information problem is inherited here: inefficient quality allocation and full rent extraction at the lowest taste, no quality distortion at the highest taste.

It appears that q^* is strictly increasing, thus separating tastes on (t', t''). However, it may not actually separate the tastes in our context. It indicates that q^* is separating only if the buyer of the tastes is wealthy enough to purchase the quality designed for him. In the opposite case, he is obliged to choose a quality within his budget. As seen in Proposition 3, the unwealthy buyer of two different tastes and the same budget, (t, w) and $(\tilde{t}, w)$, chooses the same quality and pays the same price w. Hence bunching.

Compared to the last section where the seller proposes the highest quality or none, faced to budget-constrained buyers, the seller offers a wide range of quality. By doing so, he tries to sell the highest quality within the buyers' budget and make them pay the highest possible price. The result in the last section of no discrimination of buyers' tastes lead us to conclude that the sorting of the tastes obtained in this section is ascribed wholly to the seller's consideration of the buyers' limited budget. The budget constraint is, thus, another motive of discrimination besides the standard motive in adverse selection literature.

6 Comparison with the strong incentive compatibility approach

In this section, instead of solving the seller's maximisation programme, we compare the approach of the weak mechanism and that of the strong mechanism. It should be emphasised that the results of this section do not rely on any regularity assumptions on g. They hold whether or not g is continuous not to mention all the other assumptions upon g and G.

Let us define the two-dimensional mechanism in taste and budget. To distinguish clearly a one-dimensional mechanism in the weak approach and a full two-dimensional mechanism, we name the following map the *strong mechanism*:

$$(q(t,w),p(t,w)) : T \times W \to Q \times R.$$

Truthful revelation on the part of buyers requires that the strong mechanism should satisfy the following condition.

Definition 3 (The strong incentive compatibility). *The strong mechanism* $(q(t,w),p(t,w))$ *is* strongly incentive compatible(IC)[5] *if and only if*

$$p(t,w) \le w \quad \text{for any } (t,w) \in T \times W, \tag{SBC}$$

$$tq(t,w) - p(t,w) \ge tq(\tilde{t},\tilde{w}) - p(\tilde{t},\tilde{w})$$
$$\text{for any } (t,w) \text{ and } (\tilde{t},\tilde{w}) \in T \times W \text{ such that } p(\tilde{t},\tilde{w}) \le w. \tag{SIC}$$

The strong mechanism assigns a quality and a price to a borrower of each taste and

[5] "S" in the tags SBC, SIC stands for strong.

budget. The requirement (SBC) is necessary for the borrower to actually purchase the designated quality: the corresponding price should be within his budget.

The participation constraint is also needed in order that the buyer honestly announcing his taste and budget will purchase the quality assigned by the IC mechanism.

$$tq(t,w) - p(t,w) \geq 0 \quad \text{on } T \times W. \tag{SIR}$$

Definition 4. *The* optimal *strong mechanism is* $(q(t,w), p(t,w))_{(t,w)\in T\times W}$ *which maximises*

$$\int_T \int_W p(t,w)g(t,w)dtdw$$
$$s.t. \text{ (SBC), (SIC) } and \text{ (SIR).}$$

Let us recall the formalism of social choice theory. A *mechanism*(see Myerson (1979)) is defined as a map from a *message* set to an allocation set and in our case the allocation set is the product of the quality and price space $Q \times R$. We have two players and only the buyer has variable type: taste and budget. Thus, the *type* set consists of $T \times W$. A *direct mechanism* is defined as a mechanism with the message set coincident with the type set; otherwise a mechanism is indirect.

Given a mechanism, the agent of each type decides upon which message to send and the mechanism assigns each message sent the quality-price pair. The so-called *revelation principle* states that the outcome of an indirect mechanism is realised by a direct mechanism. This is the justification whereby most of mechanism design literature concentrates upon the direct mechanism.

Clearly, in our context, a strong mechanism is a direct mechanism while a weak mechanism is an indirect mechanism with the message set T. By the same token, a non-linear price scheme is an indirect mechanism with the message set Q.

It follows from the contrapositive of the revelation principle that some quality-price pairs realised by an IC mechanism may not be put into effect by a WIC mechanism. We investigate when the WIC preserves the same generality as the IC mechanism.

The following is assumed throughout this section.

Assumption 6. *In this section, a weak mechanism stands for one with the price function* $p(t)$ *left-continuous.*

As seen through the definitions of WIC and IC, it is difficult to compare the strong and weak optimal mechanism from up front. We shall do it by way of a non-linear price schedule. First we show that given a strong mechanism, there exists a non-linear price schedule which brings the seller larger profits. Then we question when there exists a weak mechanism which achieves the same oucome as the non-linear price schedule. Let us recall that the outcome of any indirect mechanism can be realised by a direct(strong here) mechanism. The following proposition demonstrates the first step.

Proposition 6. *Given the strong incentive compatible mechanism* $(q(t,w), p(t,w))$*, there exists a non-linear price schedule* $\tau : Q \to R$ *such that*

$$\textit{it is continuous, strictly increasing, convex and } \tau(0) = 0, \tag{14}$$

and further for all (t, w)*,*

$$\tau(y) \geq p(t, w) \tag{15}$$

where

$$y \in \operatorname*{arg\,max}_{x \; s.t. \; \tau(x) \leq w} tx - \tau(x).$$

Proof. We follow the proof of Lemma 1 in Che and Gale (2000). Let us suppose that there is given a strong mechanism $(q(t,w), p(t,w))$ satisfying (SBC), (SIC) and (SIR). We posit the following price schedule τ.

$$\tau(x) := \max_{t' \in T} \big(t'x - (t'q(t',w) - p(t',w)) + \underline{t}q(\underline{t},\overline{w}) - p(\underline{t},\overline{w})\big).$$

Let us show that it satisfies (14). First, $t'q(t',w) - p(t',w)$ is both continuous and non-decreasing in t'; for from (SIC) it follows that for t', t such that $t' \geq t$,

$$t'q(t',w) - p(t',w) \geq t'q(t,w) - p(t,w) \geq$$
$$tq(t,w) - p(t,w) \geq tq(t',w) - p(t',w).$$

Thus max in the definition exists and obviously $\tau(0) = 0$. It is also clear that τ is strictly increasing. The so-called maximum theorem obtains that τ is continuous. It is convex as the superior envelope of the linear function. We have proved (14).

Now we turn to (15). It can be shown that type (t,w) buyer likes best $q(t,\overline{w})$, faced to the price schedule $\tau(x)$(note that we are concerned about preference but not affordability): for $x \in Q$,

$$\begin{aligned}
&\tau(x) \geq tx - (tq(t,\overline{w}) - p(t,\overline{w})) + \underline{t}q(\underline{t},\overline{w}) - p(\underline{t},\overline{w}) \\
\iff & tx - \tau(x) \leq tq(t,\overline{w}) - p(t,\overline{w}) - (\underline{t}q(\underline{t},\overline{w}) - p(\underline{t},\overline{w})) \\
&= tq(t,\overline{w}) - \tau(q(t,\overline{w})).
\end{aligned}$$

Let us show (15). Suppose that (t,w) is given. (1) If $\tau(q(t,\overline{w})) \leq w$, what we have just shown leads to (15). (2) If $\tau(q(t,\overline{w})) > w$, there is x^* such that $\tau(x^*) = w$ since τ is continuous and Q is connected. In addition, we know that $tx - \tau(x)$ is concave. Accordingly, we obtain that

$$\max_{x \text{ s.t. } \tau(x) \leq w} tx - \tau(x) = tx^* - w.$$

$\square$

Note that the optimal strong mechanism attains the same outcome as the corresponding non-linear price schedule due to the direct revelation principle.

We investigate now when a WIC mechanism can replicate the quality-price allocation that the non-linear price scheme (superior to the strong mechanism in profits) puts into effect.

Let us define the maximiser of utility, given the continuous price shedule $\tau(x)$,

$$Q(t) := \arg\max_{x \in Q} tx - \tau(x). \tag{16}$$

As will be seen, if $Q(t)$ reduces to a single valued map, the matter is simplified but in general, $Q(t)$ is not necessarily single-valued. We need a few lemmas.

Lemma 4. *If τ is continuous, $Q(t)$ is upper semicontinuous*[6] *as a multi-valued map.*

Proof. See Th.6, p.53 in Aubin and Cellina (1984). □

Lemma 5. *Given a non-linear price schedule τ satisfying* (14)*, it holds good that*[7]

$$Q(T) = [0, \max Q(\bar{t})].$$

Proof. $Q(t)$ is upper semicontinuous from Lemma 4 and closed-valued due to the continuity of τ. It follows, thus, from Propositioin 1.4.8 in Aubin and Frankowska (1990) that the graph is closed. In consequence, $Q(T)$ is also closed.

Notice that whatever selection we take, i.e. $q(t) \in Q(t)$, it is non-decreasing: for, by definition, we have

$$t'q(t') - \tau(q(t')) \geq t'q(t) - \tau(q(t)),$$
$$tq(t) - \tau(q(t)) \geq tq(t') - \tau(q(t')),$$

and adding them side by side, we obtain that $q(t)$ is non-decreasing. As a result, it follows that $Q(T) \subset [0, \max Q(\bar{t})]$.

We prove the lemma by contradiction. Suppose that there exists q such that $q \in [0, \max Q(\bar{t})]$ and that $q \notin Q(T)$. Let us define $L = \{x | x < q, x \in Q(T)\}$ and $U = \{x | q < x, x \in Q(T)\}$, which are both closed since $[0, q] \cap Q(T)$ and $[q, 1] \cap Q(T)$ are closed.

[6] See the appendix for the definition.

[7] Given a set A, $\max A$ intends the maximum element.

Therefore, there exist $\max L$ and $\min U$. Due to Lemma 7 in the appendix, there exist $t_L = \max\{t|Q(t) \ni \max L\}$ and $t_U = \min\{t|Q(t) \ni \min U\}$.

It must hold that $t_L = t_U$. Suppose to the contrary and then there is $t_L < t' < t_U$. It follows from the monotonicity of $Q(\cdot)$ that $\max L < q(t') < \min U$ for any $q(t') \in Q(t')$. Then we have either $\max L < q(t') < q$ or $q < q(t') < \min U$, which is contradictory to the definition of t_L and t_U.

Let $t_L = t_U$ hold now. Then, it follows that $Q(t_L) \ni \max L$ and $Q(t_L) \ni \min U$. The maximand in (16) is concave so that $Q(t_L) \supset [\max L, \min U]$. As a result, $Q(t_L) \ni q$, which is a contradiction.

□

Now we show that there is a multi-valued map $Q(t)$ which brings the same outcome as $\tau(x)$.

Proposition 7. *Given a non-linear price schedule τ satisfying* (14)*, we obtain that*

$$\operatorname*{arg\,max}_{x \; s.t. \; \tau(x) \leq w, x \in Q} \quad tx - \tau(x) \tag{17}$$

is identical to

$$\operatorname*{arg\,max}_{q(t') \; s.t. \; \tau(q(t')) \leq w, q(t') \in Q(T)} \quad tq(t') - \tau(q(t')). \tag{18}$$

Proof. From Lemma 5, $Q(T) = [0, \max Q(\bar{t})]$. Then, even if $\{x|\tau(x) \leq w, \, x \in Q\}$ is strictly larger than $\{x|\tau(x) \leq w, \, x \in Q(T)\}$, $\arg\max$ in (17) does not contain an element outside $Q(T)$ by the definition. Now the proposition is obvious.

□

Due to the proposition, if we allow a multi-valued map as a weak mechanism, i.e. $(K, P) : T \mapsto \mathcal{P}(Q \times R)$[8], we can achieve the outcome of τ with this extended mechanism by posing

[8] $\mathcal{P}$ indicates the power set.

$$(K, P)(t) := (k(t), \tau(k(t)))_{k(t) \in Q(t)}.$$

Faced to $\tau(x)$ and $Q(t)$, the buyer makes the same quality choice and pays the same price. From the definition of $Q(t)$ in the proposition, it is evident that this multi-valued mechanism satisfies (WIC).[9]

The realisation of the outcome of τ by a WIC mechanism(single valued) is a little complex. We show the following proposition.

Proposition 8. *Under the assumption of Proposition 7, if there is a continuous selection* $q(t) \in Q(t)$*, it holds good that*

$$\operatorname*{arg\,max}_{x \; s.t. \; \tau(x) \le w, x \in Q} \quad tx - \tau(x)$$

is identical to

$$\operatorname*{arg\,max}_{q(t') \; s.t. \; \tau(q(t')) \le w, q(t') \in q(T)} \quad tq(t') - \tau(q(t')).$$

Proof. Let us show that $q(T) = [0, q(\bar{t})]$. Note that $q(T) \subset [0, q(\bar{t})]$ because q(t) is non-decreasing(see the proof of Lemma 5) and $q(0) = 0$ by the definition of $Q(t)$. Now the result is obvious since $q(T)$ is connected. The rest is identical to the proof of Proposition 7.

□

To construct a WIC mechanism which achieves the outcome by τ, we simply put

$$(q(t), p(t)) := (q(t), \tau(q(t))).$$

By construction, this weak mechanism satisfies (WIC).

[9] Here, it is intended by this that any selection(thus, single-valued weak mechanism), $(p(t), q(t)) \in (K, P)(t)$ satisfies (WIC).

In general, $Q(t)$ is not assured to be single-valued and an upper semicontinuous map does not always possess a continuous selection. Naturally, when $Q(t)$ reduces to a single-valued map, the upper semicontinuity amounts to the continuity of a single valued map and the price shedule is replicated by that map. This arises, for instance, when the maximand in (16) is strictly concave.

7 Conclusion

This article has analysed the optimal selling mechanism of the monopolist faced to the budget-constrained buyer. Multi-dimensional mechanism design involves technical difficulties. To circumvent them, we have taken an approach of reducing the dimension by virtue of an indirect mechanism. We have investigated the effectiveness and limit of the approach.

8 Appendix

8.1 The proof of Proposition 5

First we search for the solution of the relaxed problem and then verify the satisfaction of (12). First, we obtain

Lemma 6. $\lambda \leq 0$. *In particular* $\lambda(0) < 0$.

Proof. Notice that $\dot{\lambda} = G(t, \overline{w}) - G(t, tq^*(t) - u^*(t)) > 0$ a.e; for $0 \leq q \leq 1$ and $u^*(t) = \int_0^t q^*(s)ds$ so that $tq^*(t) - u^*(t) < \overline{w}$ except at $t = \bar{t}$. Therefore λ is strictly increasing. From the transversality condition, the lemma is obvious. $\square$

Let us consider the maximisation of the Hamiltonian. Since the Hamiltonian is concave, if $\frac{\partial H}{\partial q}(t, 0, u^*(t), \lambda(t)) \leq 0$, the Hamiltonian is maximised at $q = 0$ and if $\frac{\partial H}{\partial q}(t, 1, u^*(t), \lambda(t)) \geq 0$, it is maximised at $q = 1$.

We have

$$\eta(t) := \frac{\partial H}{\partial q}(t, 0, u^*(t), \lambda(t)) = tG(t, \overline{w}) + \lambda(t).$$

This is continuous and strictly increasing; for $tG(t, \overline{w})$ is non-decreasing by Assumption 4 and $\lambda(t)$ is strictly increasing. To see $\lambda(t)$ strictly increasing, note that $\dot{\lambda}(t) = G(t, \overline{w}) - G(t, tq^*(t) - u^*(t)) > 0$ a.e. because $\overline{w} > tq^*(t) - u^*(t)$ except at $t = \bar{t}$.

We have $\eta(\bar{t}) > 0$ and also $\eta(0) = \lambda(0) < 0$ from Lemma 6. Thus, there is the unique t' such that $0 < t' < \bar{t}$ and $\eta(t') = 0$. Besides, $\eta(t) < 0$ on the interval $[0, t')$. It follows that q^* is zero on $[0, t']$ and that the optimal q^* is unique on $[0, t')$. It turns out that at t' likewise, q^* is uniquely determined to be zero. This is evident because from $u^*(t') = 0$(due to absolute continuity) and $\eta(t') = 0$, we obtain that except at $q = 0$,

$$\frac{\partial H}{\partial q}(t', q, u^*(t'), \lambda(t')) = -G(t', t'q)) < 0.$$

Let us turn to the latter half part. We define

$$\theta(t) := \frac{\partial H}{\partial q}(t, 1, u^*(t), \lambda(t)) = t(G(t, \overline{w}) - G(t, t - u^*(t))) + \lambda(t).$$

$\theta(t)$ is continuous and it follows from $t - u^*(t) < \bar{t} < \overline{w}$ and $g(t, w) > 0$ that

$$\theta(\bar{t}) = \bar{t}(G(\bar{t}, \overline{w}) - G(\bar{t}, \bar{t} - u^*(\bar{t}))) > 0,$$

Likewise, $\theta(0) < 0$ from Lemma 6. There exists, thus, t'' such that $\theta(t'') = 0$.

Let us show that t'' is unique. There exists $\dot{\theta}(t)$ almost everywhere and

$$\dot{\theta}(t) = G(t, \overline{w}) - G(t, t - u^*(t)) + tG_1(t, \overline{w}) - tG_1(t, t - u^*(t)) + \dot{\lambda}(t);$$

for $1 - \dot{u}^*(t) = 0$ since $\dot{u}(t) = q$ a.e. and we have assumed $q = 1$ by the definition of $\theta(t)$.

From Assumption 5, we see that $\dot{\theta}(t) - \dot{\lambda}(t) \geq 0$ and we also know that $\dot{\lambda} = G(t, \overline{w}) -$

$G(t, tq^* - u^*) > 0$ a.e, because $g(t, w) > 0$ and $\overline{w} > tq^* - u^*$ except at $t = \bar{t}$. Consequently, θ is strictly increasing and t'' is unique.

It follows that $q^* = 1$ on $[t'', \bar{t}]$ and at the same time that the optimal q^* is unique on $(t'', \bar{t}]$. It can be shown that q^* is unique at t'' likewise. For that purpose, let us examine the left differential with respect to q of $\frac{\partial H}{\partial q}(t'', q, u^*(t''), \lambda(t''))$, estimated at the point $q = 1$, which is $\lim_{q \to 1-} -t'' g(t'', t''q - u^*(t''))$.[10] This value depends upon $t''q - u^*(t'')$. Now, notice that $u^*(t) < t$; for $\dot{u} = q$ a.e. and $0 \leq q \leq 1$ and $q^* = 0$ on $[0, t']$. As a result, the left differential is strictly negative. This proves that q^* is unique at t'' and equals 1.

We have seen that for $t \in (t', t'')$,

$$\frac{\partial H}{\partial q}(t, q, u^*(t), \lambda(t)) = \begin{cases} tG(t, \overline{w}) + \lambda(t) > 0 & \text{if } q = 0, \\ t(G(t, \overline{w}) - G(t, t - u^*(t))) + \lambda(t) < 0 & \text{if } q = 1. \end{cases}$$

Since $\frac{\partial H}{\partial q}(t, q, u^*(t), \lambda(t))$ is continuous and non-increasing in $q \in Q$, there exists an optimal $\hat{q}(t) \in (0, 1)$ such that for all $t \in (t', t'')$, $\frac{\partial H}{\partial q}(t, \hat{q}(t), u^*(t), \lambda(t)) = 0$.

$\hat{q}$ is unique on (t', t''). Note that $t\hat{q}(t) - u^*(t) > 0$ on (t', t''); for otherwise $\frac{\partial H}{\partial q}(t, \hat{q}(t), u^*(t), \lambda(t)) = tG(t, \overline{w}) + \lambda(t) > 0$. There exists therefore $\frac{\partial^2 H}{\partial q^2}(t, \hat{q}(t), u^*(t), \lambda(t))$, which is equal to $-tg(t\hat{q}(t) - u^*(t), t) < 0$. Uniqueness has been proved.

Now we set to verifying that the solution of the relaxed problem satisfies the condition ignored of the original problem.

Proposition 9. *$q^*(t)$ is continuous and non-decreasing on $[0, \bar{t}]$. Further, it is strictly increasing and differentiable on (t', t'').*

Proof. As for consinuity, Proposition 5 shows that $\hat{q}$ uniquely maximises the Hamiltonian on (t', t''). Thus, we can apply Theorem 6.1 of Fleming and Richel (1975)(see page 75).

[10] $q \to 1-$ means q converges to 1 from below.

We will show that q^* is differentiable and $\dot{q}^* > 0$ on (t', t''). Let us look upon $J(t,q) = t(G(t,\overline{w}) - G(t, tq - u^*(t))) + \lambda(t)$ as a function on $(t', t'') \times (0,1)$.

Now let us show that $J(t,q)$ is differentiable at $(t, \hat{q}(t))$ for $t \in (t', t'')$: $g(t,w)$ has a jump at $(t,w) = (t,0)$ and therefore $G(t,w)$ may not be differentiable at the point. From Proposition 5, we know $J(t, q^*(t)) = J(t, \hat{q}(t)) = 0$ on (t', t''). First, notice that $t\hat{q}(t) - u^*(t) > 0$ on $t \in (t', t'')$; for otherwise, $J(t, \hat{q}(t)) = tG(t, \overline{w}) + \lambda(t) > 0$ from Proposition 5 but this is a contradiction to the definition of $\hat{q}(t)$. Now that we have learnt that $t\hat{q}(t) - u^*(t) > 0$ on (t', t''), $J(t,q)$ is continuously differentiable in q at the point $(t, \hat{q}(t))$ when $t \in (t', t'')$. Let us turn to the differentiability of $J(t,q)$ with respect to t at $(t, \hat{q}(t))$. Since $\dot{\lambda} = G(t, \overline{w}) - G(t, tq^* - u^*)$ almost everywhere and q^* is continuous, λ is continuously differentiable on (t', t''); recall also that $0 < tq^*(t) - u^*(t)$ on (t', t'') above. Therefore, $J(t,q)$ is continuously differentiable with respect to t at $(t, \hat{q}(t))$ when $t \in (t', t'')$. Combining the two results, we obtain that $J(t,q)$ is differentiable at $(t, \hat{q}(t))$ for $t \in (t', t'')$. In addition, $\frac{\partial \tilde{J}}{\partial q}(t, \hat{q}(t)) \neq 0$ when $t \in (t', t'')$.

Now we can apply an implicit function theorem (see Theorem 3.8.2 Schwartz (1997)) and conclude that q^* is differentiable on (t', t''). $\dot{q}^*$ is expressed as

$$\dot{q}^*(t) = -\frac{\dfrac{\partial J}{\partial t}(t, q^*)}{\dfrac{\partial J}{\partial q}(t, q^*)} \quad \text{for } t \in (t', t'').$$

As was seen, $\frac{u^*(t)}{t} < q^* < 1$ on (t', t'') so that for $t \in (t', t'')$,

$$\frac{\partial J}{\partial q}(t, q^*) = -t^2 g(t, tq^* - u^*(t)) < 0.$$

On the other hand, for $t \in (t', t'')$,

$$\begin{aligned}\frac{\partial J}{\partial t}(t,q^*) =& G(t,\overline{w}) - g(t,tq^* - u^*(t)) + tG_1(t,\overline{w}) \\ & - tg(t,tq^* - u^*(t))(q^* - \dot{u}^*(t)) - tG_1(t,tq^* - u^*(t)) + \dot{\lambda}(t).\end{aligned}$$

Notice that u is everywhere differentiable because q is continuous and therefore $\dot{\lambda}(t)$ always exists on (t', t''). From (11) it follows that

$$g(t,tq^* - u^*(t))(q^* - \dot{u}^*(t)) = 0.$$

From Assumption 5, we have for $t \in (t', t'')$

$$G(t,\overline{w}) - g(t,tq^* - u^*(t)) + tG_1(t,\overline{w}) - tG_1(t,tq^* - u^*(t)) \geq 0.$$

We also know that $\dot{\lambda} > 0$ on (t', t'') for $\dot{\lambda} = G(t,\overline{w}) - G(t,tq^* - u^*)$ and $\overline{w} > tq^* - u^*$ for $t \in (t', t'')$.

We have deduced that

$$\frac{\partial \tilde{J}}{\partial t}(t,q^*) > 0.$$

It is established that $\dot{q}^*$ is strictly increasing on (t', t'').

□

8.2 On upper semicontinuity

Definition 5. *Suppose that X and Y are topological spaces. A multi-valued map F from X to the subsets of Y is upper semicontinuous if for any neighbourhood V of $F(x)$, there exists a neighbourhood U of x such that $F(U) \subset V$.*

Lemma 7. *Suppose that X and Y are topological spaces and that F is a multi-valued map from X to the subsets of Y. F is upper semicontinuous if and only if the inverse image of any closed set A, i.e. $\{x | F(x) \cap A \neq \emptyset\}$ is closed.*

Proof. See Proposition 1.4.4, p.40 in Aubin and Frankowska (1990). □

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